MATHEMATICAL DESCRIPTION OF THE PROCESS OF FILTRATIONAL FLUSHING OF SEDIMENTS IN A REGIME CLOSE TO IDEAL MIXING

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An asymptotic analysis of the solution of the equations of filtrational flushing of sediments for low values of the Peclet number is performed.

Filtrational flushing of sediments, i.e., removal of impurities from a porous body by injecting a flushing liquid through it, is a mass-exchange process. At the present time, the basic model of this process is a model in which two intercoupled concentrations of the impurity are introduced: in the flow-through part and in the stagnation zones [1]. According to [1], the equations for determining these concentrations have the form:

$$\mathrm{Pe}\,(\partial c_1/\partial\tau + \partial c_1/\partial z) = \partial^2 c_1/\partial z^2 + n\,(c_2 - c_1), \tag{1}$$

$$\mathrm{Pe}\,\varphi\partial c_2/\partial\tau = n\,(c_1 - c_2), \tag{2}$$

where

$$\tau = tu/l\varepsilon, \quad n = kl^2/D, \quad \varphi = (1-\varepsilon)/\varepsilon.$$

Systems of equations analogous to (1)-(2) arise in the theory of heat conduction in heterogeneous media [2], adsorption, and elsewhere. It is known [3] that the coefficients of the model D and k may be assumed to be constants only when the relaxation of the interphase exchange is completed. It is precisely these (long) time intervals which are of interest for the description of the flushing process. Since at the present time only the cases of constant D and k find application in the theory of flushing, which is a result of the inadequate study of the time dependence of these coefficients and the difficulty of their experimental determination [4], we shall assume that these coefficients are constant. We also note that Eqs. (1)-(2) ignore the adsorption of the impurity on the interphase surface which sometimes occurs.

The boundary conditions for Eq. (1) will be as follows:

$$\left.\frac{\partial c_1}{\partial z}\right|_{z=0} = \mathrm{Pe}\,c_1, \quad \left.\frac{\partial c_1}{\partial z}\right|_{z=1} = 0. \tag{3}$$

The first condition expresses the absence of an impurity flow at the input to the porous body and the second (at the output) is the usual Dankwerst condition. Initially the concentration of the impurity can have any distribution in the porous body. Therefore, the initial conditions will be as follows:

$$c_1|_{\tau=0} = K(z), \quad c_2|_{\tau=0} = L(z). \tag{4}$$

We shall assume that the number Pe is small. Then, to find the solutions of the problem we can use the methods of the theory of small perturbations [5]. We represent the solution sought to the problem (1)-(4) in the form of the following expansions in powers of Pe:

$$c_1 = f_0(z, \tau) + \mathrm{Pe}\,f_1(z, \tau) + \ldots, \quad c_2 = F_0(z, \tau) + \mathrm{Pe}\,F_1(z, \tau) + \ldots \tag{5}$$

Substituting the expansions (5) into Eqs. (1) and (2), as well as into the boundary conditions (3), we obtain, by grouping terms of the same order with respect to the Pe number, the following chain of equations and boundary conditions:

$$\partial^2 f_0/\partial z^2 = n\,(f_0 - F_0), \quad f_0 = F_0, \quad \partial f_0/\partial z|_{z=0;\,1} = 0, \tag{6}$$

$$\partial^2 f_1/\partial z^2 + n\,(F_1 - f_1) = \partial f_0/\partial\tau + \partial f_0/\partial z, \quad \partial f_1/\partial z|_{z=0} = f_0,$$

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$$\varphi\partial F_0/\partial\tau = n(f_1 - F_1), \quad \partial f_1/\partial z|_{z=1} = 0, \tag{7}$$

$$\partial^2 f_2/\partial z^2 + n(F_2 - f_2) = \partial f_1/\partial\tau + \partial f_1/\partial z,$$
$$\varphi\partial F_1/\partial\tau = n(f_2 - F_2), \quad \partial f_2/\partial z|_{z=0} = f_1, \quad \partial f_2/\partial z|_{z=1} = 0 \tag{8}$$

and so on. We shall only need the Eqs. (6)-(8).

Equations (6)-(8) do not contain time derivatives of the functions sought. (Equations (7) and (8) contain time derivatives of the functions which must be determined beforehand.) Therefore, the initial conditions (4) must be dropped. The problem is a singular-perturbation problem [5], and some properties of the "inner" solution are required to solve it. The problem (6) has the following solution:

$$f_0 = F_0 = A(\tau), \tag{9}$$

where $A(\tau)$ is an unknown function of time. We substitute this solution into the equations and boundary conditions (7). Carrying out the integration, we find

$$f_1 = (1+\varphi)z^2 A'(\tau)/2 + zB(\tau) + C(\tau), \quad F_1 = f_1 - \varphi A'/n, \tag{10}$$

where $B(\tau)$ and $C(\tau)$ are unknown functions of time. Substitution of the expressions (10) into the boundary conditions leads to the relations

$$B = A, \quad (1+\varphi)A'(\tau) + A(\tau) = 0. \tag{11}$$

We have thereby obtained a differential equation for the function $A(\tau)$. Proceeding analogously, we find that the function $C(\tau)$ must satisfy the following equation:

$$(1+\varphi)C'(\tau) + C(\tau) + A(\tau)/6 = 0. \tag{12}$$

The initial conditions which are required to determine the functions $A(\tau)$ and $C(\tau)$ uniquely must be established by joining with the inner solution. We now proceed to determine some of its characteristics.

It is easy to see that $T = \tau/\mathrm{Pe}$ is a suitable inner variable. Equations (1) and (2) then assume the form:

$$\partial c_1/\partial T + \mathrm{Pe}\,\partial c_1/\partial z = \partial^2 c_1/\partial z^2 + n(c_2 - c_1), \tag{13}$$

$$\varphi\partial c_2/\partial T = n(c_1 - c_2). \tag{14}$$

The starting conditions (3) and (4) will be additional conditions for the inner problem. We seek the solution of Eqs. (13) and (14) in the form of expansions:

$$c_1 = g_0(z, T) + \mathrm{Pe}\,g_1(z, T) + \ldots, \quad c_2 = G_0(z, T) + \mathrm{Pe}\,G_1(z, T) + \ldots. \tag{15}$$

Substituting the expansions (15) into Eqs. (13) and (14) as well as into the additional conditions (3) and (4), we obtain in the zeroth order approximation with respect to the Pe number the following problem:

$$\partial g_0/\partial T = \partial^2 g_0/\partial z^2 + n(G_0 - g_0),$$
$$\varphi\partial G_0/\partial T = n(g_0 - G_0), \tag{16}$$

$$\left.\frac{\partial g_0}{\partial z}\right|_{z=0;\,1} = 0, \quad g_0|_{T=0} = K(z), \; G_0|_{T=0} = L(z). \tag{17}$$

The solution of the problem (16) and (17) describes the process of flushing with short times (T of the order of unity). Strictly speaking, this is the solution for "ideal mixing," because it determines the rapid ($\tau \sim \mathrm{Pe}$) equalization of the concentrations by diffusion, while the argument of the outer solution (the function $A(\tau)$ in the zeroth-order approximation) is the "slow" time with the characteristic scale l/u. In practice, for the sediment-flushing process, the behavior of the solution at long times, i.e., the outer solution, is of basic interest. We shall therefore not study here in detail the solution of the problem (16) and (17), but we shall only find the asymptotic behavior of g_0 in the limit $T \to \infty$ required for joining with the outer solution.

We apply to Eqs. (16) and the boundary conditions (17) Laplace's transformation with respect to the variable T. We shall denote the quantities in the transform space by a bar. After some calculations, we find that the function $\overline{g_0}$ satisfies the equation

$$d^2\overline{g}_0/dz^2 = \psi^2\overline{g}_0 - J(z, p) \tag{18}$$

and the boundary conditions $d\bar{g}_0/dz = 0$ at $z = 0$ and $z = 1$. Here the functions ψ and J are defined by the expressions

$$\psi^2 = n + p - n^2/(\varphi p + n), \qquad J = K(z) + n\varphi L(z)/(\varphi p + n). \tag{19}$$

The solution of Eq. (18) with the formulated boundary conditions has the form

$$\bar{g}_0 = \frac{\operatorname{ch}(\psi z)}{\psi \operatorname{sh}\psi} \int_0^1 \operatorname{ch}[\psi(1-\xi)]\, J(\xi, p)\, d\xi - \int_0^z \frac{\operatorname{sh}[\psi(z-\xi)]}{\psi} J(\xi, p)\, d\xi. \tag{20}$$

As is well known [6], the limit g_0 as $T \to \infty$ can be determined from the following equality:

$$\lim_{T\to\infty} g_0(z, T) = \lim_{p\to 0} p\bar{g}_0(z, p) \tag{21}$$

under the condition that the second limit exists. It is evident from Eqs. (19) that in the limit $p \to 0$ J is of the order of unity and $\psi^2 \sim (\varphi+1)p$. Therefore, only the first term will contribute to the limit (21). From here we find, using the principle of asymptotic joining [5],

$$A(0) = f_0(z, 0) = g_0(z, \infty) = \lim_{p\to 0} \frac{p}{\psi \operatorname{sh}\psi} \int_0^1 \operatorname{ch}[\psi(1-\xi)]\, J(\xi, p)\, d\xi = \frac{1}{(\varphi+1)} \int_0^1 [K(z) + \varphi L(z)]\, dz, \tag{22}$$

the initial condition sought for the function A. We can now integrate Eq. (11). We obtain

$$A(\tau) = \exp(-tu/l) \int_0^1 [\varepsilon K(z) + (1-\varepsilon) L(z)]\, dz. \tag{23}$$

In Eq. (23), we have transformed to the dimensional time t and the quantity ε in order to show more clearly the characteristic time scale and the role of the volume fractions ε and $(1 - \varepsilon)$ in the zeroth-order solution with respect to Pe. We can see that in the outer solution the initial conditions (4) have been transformed into some integral characteristics; in addition, the functions K(z) and L(z) enter into it with the corresponding volume fractions ε and $(1 - \varepsilon)$. We also note that the zeroth-order approximation (23) does not contain the parameter n. It will affect the process in the outer solution only in the first approximation with respect to Pe (Eq. (29)).

The equations in the first-order approximation with respect to Pe in the inner expansion will have the form

$$\partial g_1/\partial T + \partial g_0/\partial z = \partial^2 g_1/\partial z^2 + n(G_1 - g_1),$$
$$\varphi \partial G_1/\partial T = n(g_1 - G_1). \tag{24}$$

The initial conditions for Eqs. (24) will be zero conditions, and the boundary conditions will assume the following form:

$$\partial g_1/\partial z\big|_{z=0} = g_0, \quad \partial g_1/\partial z\big|_{z=1} = 0. \tag{25}$$

The limiting relation (21) leads to the conclusion that it is convenient to join the inner and outer expansions in the region of the Laplace transformations of the corresponding functions. Application of the Laplace transformation to the problem (24) and (25) leads to an equation analogous to (18), from whose solution the expansion of the function $\bar{g}_1$ in powers of p is found:

$$\bar{g}_1 \sim A'(0)/p^2 + [zA(0) - z^2 A(0)/2 - A(0)/6 + \frac{(1-\varepsilon)^2}{n} \int_0^1 L(z)\, dz - \frac{2A(0)(1-\varepsilon)^2}{n} - \frac{\varepsilon}{2} \int_0^1 \xi^2 J(\xi, 0)\, d\xi]/p + \ldots \tag{26}$$

We now rewrite the outer expansion of c_1 in terms of inner variables and we expand in powers of Pe:

$$c_1 \sim f_0(T\,\mathrm{Pe}) + \mathrm{Pe}\, f_1(T\,\mathrm{Pe}) \sim f_0(0) + \mathrm{Pe}\,[f_1(0) + T\partial f_0/\partial\tau|_{\tau=0}] + \ldots \tag{27}$$

The application of Laplace's transformation to the expansion (27) gives

$$\bar{c}_1 \sim A(0)/p + \mathrm{Pe}\,[f_1(z, 0)/p + A'(0)/p^2] + \ldots \tag{28}$$

Comparison of the expansions (26) and (28) leads to the initial condition sought for the function C:

$$C(0)=\frac{(1-\varepsilon)^2}{n}\int_0^1 L(z)\,dz-\frac{2A(0)(1-\varepsilon)^2}{n}-\frac{A(0)}{6}-\frac{\varepsilon}{2}\int_0^1 \xi^2 J(\xi,\ 0)\,d\xi. \tag{29}$$

Now the solution of Eq. (12) is easily found:

$$C(\tau)=\exp(-\tau\varepsilon)[C(0)-\varepsilon A(0)\tau/6], \tag{30}$$

where the quantities A(0) and C(0) are determined by Eqs. (22) and (29). As before, only the integral characteristics of the functions K(z) and L(z) affect the behavior of the outer solution. The expression for the concentration c_1 with accuracy up to terms of the order of Pe can be written in the form

$$c_1=\exp(-\tau\varepsilon)\{A(0)[1+\mathrm{Pe}(z-z^2/2-\varepsilon\tau/6)]+\mathrm{Pe}\,C(0)\}+\dots \tag{31}$$

By making the transformation τ: $\tau\to\tau(1+\mathrm{Pe}/6)$, the secular term $-\tau\varepsilon A(0)\mathrm{Pe}/6$ can be eliminated in the usual manner [5], thereby expanding the region of applicability of the expression (31). We have

$$c_1=\exp[-\tau\varepsilon(1+\mathrm{Pe}/6)]\{A(0)[1+\mathrm{Pe}(z-z^2/2)]+\mathrm{Pe}\,C(0)\}+O(\mathrm{Pe}^2). \tag{32}$$

It should be noted, however, that the improvement of the expansion (31) occurred only for quite large values of τ. In the limit $\tau\to 0$, Eqs. (31) and (32) will give an error, which can be eliminated by invoking the inner solution. Thus the magnitude of the correction $O(\mathrm{Pe}^2)$ in relation (32) is irregular in the limit $\tau\to 0$.

As an illustration of the results obtained, we shall study the case of greatest practical interest when K(z) = L(z) = 1. Calculating the quadratures required in Eqs. (22) and (29), we obtain from relation (32) the following expression for the concentration of the impurity in the flow-through part:

$$c_1=\exp[-\tau\varepsilon(1+\mathrm{Pe}/6)]\{1+\mathrm{Pe}[z-z^2/2-1/3-(1-\varepsilon)^2/n]\}+\dots \tag{33}$$

A method for reducing the system (1) and (2) to a single "equivalent" equation, from which it is possible to obtain the approximate solutions of problems for sufficiently long times, was proposed in [3]. It is of interest to compare the result (33) with the solution obtained from the equivalent equation, which in our variables will have the form:

$$\mathrm{Pe}\left(\frac{1}{\varepsilon}\frac{\partial c_1}{\partial\tau}+\frac{\partial c_1}{\partial z}\right)=\frac{\partial^2 c_1}{\partial z^2}+n\sum_{k=2}^{\infty}(-1)^k\left(\frac{\varphi\,\mathrm{Pe}}{n}\right)^k\frac{\partial^k c_1}{\partial\tau^k}. \tag{34}$$

The solution with an accuracy up to terms of the order of Pe can be obtained by dropping all terms in the sum. In this case, it is clear that the equation obtained will not describe exchange between the flow-through part and the stagnation zones, since the parameter n drops out of the problem, whereas the relation (33) contains this parameter. Because the solution of the equation

$$\mathrm{Pe}[\partial c_1/\partial(\varepsilon\tau)+\partial c_1/\partial z]=\partial^2 c_1/\partial z^2 \tag{35}$$

is of interest in itself for the description of a number of mass-transfer processes (including also for flushing sediments), we shall present its solution with an accuracy up to terms of the order of Pe. Following the scheme presented above, it is easy to verify that the equations of order O(1) and O(Pe) will coincide with Eqs. (11) and (12). The inner solution, however, found by the usual methods of mathematical physics gives

$$c_1=1+\mathrm{Pe}\left[z-\frac{z^2}{2}-\frac{1}{3}-\varepsilon T+\frac{2}{\pi^2}\sum_{k=1}^{\infty}\frac{(-1)^k}{k^2}\exp(-\pi^2k^2\varepsilon T)\cos(\pi kz)\right]+\dots \tag{36}$$

Dropping the terms which are exponentially small in the limit $T\to\infty$ and joining with the outer solution, we obtain the initial conditions for Eqs. (11) and (12)

$$A(0)=1,\quad C(0)=-1/3. \tag{37}$$

The solution of Eqs. (11) and (12) with the conditions (37) permits writing the solution of Eq. (35) in the form

$$c_1=\exp[-\tau\varepsilon(1+\mathrm{Pe}/6)]\{1+\mathrm{Pe}(6z-3z^2-2)/6\}+\dots, \tag{38}$$

where we have eliminated, analogously to the relation (31), the secular term. The expression (38) raises the accuracy of the well-known relation of H. Brenner for "ideal mixing" [7] up to terms of order Pe. Using the inner solution, we construct the uniformly applicable expansion of the first order

$$c_1 = \exp[-\tau\varepsilon(1 + \mathrm{Pe}/6)]\{1 + \mathrm{Pe}(6z - 3z^2 - 2)/6\} + \frac{2\mathrm{Pe}}{\pi^2}\sum_{k=1}^{\infty}\frac{(-1)^k}{k^2}\exp(-\pi^2k^2\varepsilon\tau/\mathrm{Pe})\cos(\pi kz) + O(\mathrm{Pe}^2). \quad (39)$$

If one or several terms are retained in the sum of Eq. (34), then the parameter n will enter into the equation, but a calculation, analogous to the one performed above, shows that the outer solution of the problem remains unchanged — (38). Transfer between the flow-through part and the stagnation zones will only affect the inner solution, and then the transfer parameter n will occur only in the exponent in a sum of the type (36). This leads to the conditions (37) and the same outer solution up to terms of order Pe. The reason for the incommensurability of the solution (38) with Eq. (33) lies in the fact that the expression for c_2

$$c_2 = \sum_{k=0}^{\infty}(-1)^k\left(\frac{\varphi\mathrm{Pe}}{n}\right)^k\frac{\partial^k c_1}{\partial\tau^k}, \quad (40)$$

which we used in analogy to [3] in the method of the equivalent equation, is applicable only for long times (as is the equivalent-equation method in general). At short times, however, using the operational method, we find from Eq. (2), using the initial condition L(z) = 1, that the term $\exp(-n\tau/\varphi\mathrm{Pe})$, which significantly affects the behavior of the inner solution and, therefore, the joining with the outer solution, must be added to the sum (40). An important fact here is that in order to describe correctly the inner solution it is necessary to retain all terms in the sum (34) in addition to the exponential term mentioned above.

The considerations presented above for the example of the problem with constant D and k show the significant influence of the starting stage of the process on the behavior of the solution at long times. We also note that when a sufficient number of terms are retained in the equivalent equation (34), equations of the type (11) and (12) of the outer problem, which completely coincide with the equations obtained by the previous method, can be obtained. In this case, the use of the equivalent equation leads to some simplification of the intermediate calculations.

NOTATION

c_1 and c_2, dimensionless concentrations of the impurity in the flow-through part and in the stagnation zones, respectively; D, dispersion coefficient; f and F, external variables; g and G, internal variables; k, kinetic mass-transfer coefficient; K(z), L(z), dimensionless initial values of the impurity concentrations in the flow-through part and in the stagnation zones (scaled to the characteristic value of the concentration); l, thickness of the sediment; $\mathrm{Pe} = ul/D$, Peclet number; p, Laplace transform variable; $T = \tau/\mathrm{Pe}$, inner time; t and x, dimensional time and the coordinate along the layer; u, filtration velocity; $z = x/l$, dimensionless coordinate; ε, fraction of the flow-through part in the total void volume of the porous body; and τ, outer time.

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